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Laguerre Expansions of Gel'fand–Shilov Spaces

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The subspaces G_α, G^β, and G_α^β $(\alpha, \beta \geq 0)$ of Schwartz' space S^+ in $(0, +\infty)$ are associated with the Hankel transform in the same way as the Gel'fand–Shilov spaces S_α, S^β, and S_α^β are associated with the Fourier transform. Indeed, if we consider the Hankel transform H_γ $(\gamma > -1)$ defined by $\mathscr{H}_\gamma(f)(t) = \frac{1}{2}\int_0^\infty (xt)^{-\gamma/2} x^\gamma J_\gamma(\sqrt{xt}) f(x)\, dx$ then $\mathscr{H}_\gamma$ is an isomorphism from G_α, G^β, and G_α^β onto G^α, G_β, and G_β^α, respectively. So, the spaces G_α^α are invariant for $\mathscr{H}_\gamma$. In this paper, we characterize the spaces G_α^α $(\alpha \geq 1)$ in terms of their Fourier–Laguerre coefficients. Also, we characterize the range of the Fourier–Laplace operator $\mathscr{F}_D$ defined by $\mathscr{F}_D(f)(w) = \int_0^\infty f(t)\, e^{-(1/2)((1+w)/(1-w))t}\, dt$ for $w \in D = \{w \in \mathbb{C} : |w| \leq 1\}$ when it acts on the space G_α^α.

1. INTRODUCTION

The spaces of type S (S_α, S^β, and S_α^β) were introduced by I. M. Gel'fand and G. E. Shilov to extend the Fourier transform to a class of functionals larger than that of tempered distributions. Their topological duals (tempered ultradistribution spaces) have been successfully used in Cauchy problems, in differential operator theory, and also in spectral analysis (see [GS1], [GS2]).

In order for a function f to belong to these spaces, the order in k and p of $\sup_{t \in \mathbb{R}} |t^k f^{(p)}(t)|$ is forced to be bounded by a double fixed sequence: sequences of types $C_p A^k k^{\alpha k}$, $C_k B^p p^{\beta p}$, and $C A^k B^p k^{\alpha k} p^{\beta p}$ for the spaces S_α, S^β, and S_α^β, respectively.

The spaces S_α^α are the most important of them, because as Schwartz space S, they are invariant under the Fourier transform. All these spaces have been characterized in terms of the Fourier–Hermite coefficients. Hermite orthogonal system and Fourier transform are related to each other because the functions of this orthonormal system are the eigenfunctions for the Fourier transform. Here, it is worthwhile to mention the theory of the Fourier transform developed by Korevaar from the point of view of Hermite expansions [K].

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L. Schwartz [S] proved that a function is in the space S if and only if its sequence of Fourier–Hermite coefficients is a rapidly decreasing sequence. Several authors have proved (see, e.g., [A], [EG]) that a function f belongs to S_α^α if and only if there exist two constants $c>0$, $a>1$ such that its sequence of Fourier–Hermite coefficients satisfies

$$|a_n| \leqslant c a^{-n^{1/2\alpha}} \qquad \text{for} \quad n \geqslant 0.$$

Let us consider the transform analogous to the Fourier transform for the positive real line $(0, +\infty)$, that is, the Hankel–Clifford transform $\mathscr{H}_0$ defined by

$$\mathscr{H}_0(f)(t) = \frac{1}{2} \int_0^\infty f(x)\, J_0(\sqrt{xt})\, dx,$$

and the Laguerre orthonormal system $(L_n(t)\, e^{-t/2})_n$ (here, $L_n(t)$ are the Laguerre polynomials) in $L^2(0, +\infty)$. As in the previous case, the functions $L_n(t)\, e^{-t/2}$ are eigenfunctions for the Hankel–Clifford transform.

The Hankel–Clifford transform is an isomorphism from the Schwartz space defined in $(0, +\infty)$ (denoted by S^+) onto itself, and it has been proved (see [D1], [Gu], [Z]) that as for the space S, a function in the space S^+ is characterized because its Fourier–Laguerre sequence is a rapidly decreasing one.

Recently, the author [D2] introduced the subspaces G_α, G^β, and G_α^β of the Schwartz space S^+. They are associated with the Hankel–Clifford transform in the same way as the Gel'fand–Shilov spaces are associated with the Fourier transform. Thus, the spaces G_α^α are invariant under the Hankel–Clifford transform.

A natural question arises:

QUESTION 1. *What happens with the Fourier–Laguerre coefficients in these spaces?*

Let us note that for $1 \leqslant \alpha \leqslant 2$ this question has been solved. Indeed, in [D2], we proved that the spaces G_α^α are very related to the spaces $S_{\alpha/2}^{\alpha/2}$. More precisely

$$G_\alpha^\alpha = \{ f(\sqrt{t}) : f \in S_{\alpha/2,\, \text{even}}^{\alpha/2} \}$$

(here, X_{even} denotes the subspace of the even functions which belong to X). Hence, our question is equivalent to the following: (see [EG]):

QUESTION 2. *What happens with the Fourier coefficients in the spaces* $S_{\alpha/2,\, \text{even}}^{\alpha/2}$ *with respect to the orthonormal system* $(\sqrt{2} L_n(x^2)\, e^{-x^2/2})_n$?

S. J. L. van Eijndhoven and J. de Graaf proved in [EG] that, for

$1 \leqslant \alpha \leqslant 2$, a function f belongs to the space $S^{\alpha/2}_{\alpha/2,\,\text{even}}$ if and only if there exist two constants $c > 0$, $a > 1$ such that

$$|a_n| \leqslant c a^{-n^{1/\alpha}} \qquad \text{for} \quad n \geqslant 0,$$

where $(a_n)_n$ is the Fourier sequence (with respect to $(\sqrt{2} L_n(x^2)\, e^{-x^2/2})_n$) of the function f.

Also, in [D3], the answer to Question 1 for $\alpha = 1$ can be found.

As the main result of this paper, we give a complete answer to Question 1 (and so to Question 2) (Section 3). We prove that (as for the space S^{α}_{α}) in order for a function to belong to the space G^{α}_{α} (for $\alpha \geqslant 1$) it is neccessary and sufficient that there exist two constants, $c > 0$, $a > 1$ such that its sequence of Fourier Laguerre coefficients satisfies

$$|a_n| \leqslant c a^{-n^{1/\alpha}} \qquad \text{for} \quad n \geqslant 0.$$

We extend this result for the generalized Laguerre orthonormal system and for the dual space $(G^{\alpha}_{\alpha})'$.

Two integral transforms play a fundamental role in the proof of these results. One of them is the above-mentioned Hankel–Clifford transform. The other is the Fourier–Laplace type of operator $\mathscr{F}_D$ defined by

$$\mathscr{F}_D(f)(w) = \int_0^{\infty} f(t)\, e^{-(1/2)((1+w)/(1-w))t}\, dt \qquad \text{for} \quad w \in D. \tag{1.1}$$

We characterize the range of the operator $\mathscr{F}_D$ acting in the space G^{α}_{α}.

In [D2], we stated the relation between our spaces of type G and the Gel'fand–Shilov spaces defined on $(0, +\infty)$. In particular, we proved that $G_{\alpha} = S^{+}_{\alpha}$ for $\alpha \geqslant 0$ and that $G^{\beta} \subset S^{+\beta-1}$ for $\beta \geqslant 1$. To complete this paper (Section 4), we prove (from an exhaustive study of the operator $\mathscr{F}_D$) that the above result is the best possible; that is, G^{β} is always different from the space $S^{+\beta-1}$, and $S^{+\beta-1}$ is the smallest space of type $S^{+\gamma}$, which contains the space G^{β}.

Notations. As usual $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{N}$ denote real, complex, and nonnegative integer numbers. We write $\mathbb{C}^{\mathbb{N}}$ for the space of complex sequences. Given a complex number z, its real and imaginary parts are denoted by $\Re z$ and $\Im z$, respectively. Finally, J_{μ} and I_{μ} denote the Bessel functions of the first kind and the Bessel functions of the imaginary argument, respectively.

2. Preliminaries

To begin with, we give the definitions of the spaces S^{+}, G_{α}, G^{β}, and G^{β}_{α}, for α, $\beta \geqslant 0$.

$$S^{+} = \{ f \in \mathscr{C}^{\infty}((0, +\infty)) : \forall k,\ p \geqslant 0,\ \exists C_{k,p} > 0, \\ \sup_{t > 0} |t^k f^{(p)}(t)| \leqslant C_{k,p} \},$$

$$G_\alpha=\{f\in S^+ : \exists A,\ C_p>0,\ \forall k,\ p\geqslant 0,\ \sup_{t>0}|t^{(k+p)/2}f^{(p)}(t)|\leqslant C_pA^kk^{(\alpha/2)k}\}$$

$$G^\beta=\{f\in S^+ : \exists B,\ C_k>0,\ \forall k,\ p\geqslant 0,\ \sup_{t>0}|t^{(k+p)/2}f^{(p)}(t)|\leqslant C_kB^pp^{(\beta/2)p}\}$$

$$G_\alpha^\beta=\{f\in S^+ : \exists A,\ B,\ C>0,\ \forall k,\ p\geqslant 0,\ \sup_{t>0}|t^{(k+p)/2}f^{(p)}(t)|\leqslant CA^kB^pk^{(\alpha/2)k}p^{(\beta/2)p}\}.$$

We endow the spaces G_α, G^β, and G_α^β with the obvious inductive limit topology (see Section 2 of [D2]).

We briefly comment on some properties of these spaces which we use in this paper.

Here, we introduce spaces of type G using $\|\cdot\|_\infty$ norm. In [D2], we used $\|\cdot\|_2$, but as we noted in Note 2.2(a) of [D2], both $\|\cdot\|_\infty$ and $\|\cdot\|_2$ norms give the same spaces.

As for spaces of type S (see [Ka]), we proved that $G_\alpha^\beta=G_\alpha\cap G^\beta$. Moreover, in order for $f\in G_\alpha^\beta$, it is enough that $f\in S^+$ and there exist constants A, B, $C>0$ such that

$$\begin{aligned}\|t^{k/2}f(t)\|_2&\leqslant CA^kk^{(\alpha/2)k} \quad \text{for} \quad k\geqslant 0\\ \|t^{p/2}f^{(p)}(t)\|_2&\leqslant CB^pp^{(\beta/2)p} \quad \text{for} \quad p\geqslant 0\end{aligned} \tag{2.1}$$

(see the proof of Lemma 3.4 in [D2]).

The spaces G_α, G^β are closed under differentiation and multiplication by t (see part(d) of Note 2.2 and Lemma 3.3 in [D2]). As $G_\alpha^\beta=G_\alpha\cap G^\beta$, it follows that the space G_α^β has the same property.

As we wrote in the Introduction, the spaces are associated with the Hankel–Clifford transform in the same way as the Gel'fand–Shilov spaces are associated with the Fourier transform. We state this result in the following (Theorem 3.5 of [D2]):

THEOREM A. *The Hankel–Clifford transform* $\mathscr{H}_\gamma$ $(\gamma>-1)$ *defined by*

$$\mathscr{H}_\gamma(f)(t)=\frac{1}{2}\int_0^\infty (xt)^{-\gamma/2}\,x^\gamma J_\gamma(\sqrt{xt})\,f(x)\,dx$$

is an isomorphism from the spaces G_α, G^β, *and* G_α^β *onto* G^α, G_β, *and* G_β^α, *respectively. Hence, the space* G_α^α *is invariant for this transform.* ∎

We recall that the key in the proof of Theorem A is the formula (see Lemma 3.2 of [D2])

$$\|t^{(p+k+\gamma)/2} f^{(p)}(t)\|_2 = 2^{-p+k} \|t^{(p+k+\gamma)/2} \mathscr{H}_\gamma^{(k)} f(t)\|_2, \tag{2.2}$$

which we also use in this paper.

Let us introduce the so-called fractional power of the Hankel–Clifford transform $\mathscr{I}_{z,\gamma}: L^2((0,+\infty), t^\gamma\, dt) \to L^2((0,+\infty), t^\gamma\, dt)$ defined by

$$\mathscr{I}_{z,\gamma}(f)(t) = \frac{1}{1-z} e^{-(1/2)((1+z)/(1-z))t} \times \int_0^\infty e^{-(1/2)((1+z)/(1-z))x} (xtz)^{-\gamma/2} x^\gamma I_\gamma\left(\frac{2\sqrt{xtz}}{1-z}\right) f(x)\, dx$$

(where $z \in \mathbb{C}$, $|z| = 1$, $z \neq 1$), which is an isometry from $L^2((0,+\infty), t^\gamma\, dt)$ onto itself.

The Hankel–Clifford transform can be obtained by putting $z = -1$ in this formula. In [D2, Theorem 3.5], we proved that this transform is an isomorphism from the space G_α^α onto itself. However, this transform is not so well behaved when acting on the other spaces of type G. Indeed, if we put $\Phi_z(t) = e^{(1/2)((1+z)/(1-z))t}$, then:

THEOREM B. *The fractional power of the Hankel–Clifford transform* $\mathscr{I}_{z,\gamma}$ *is an isomorphism from* G_α *onto* $\Phi_{z^{-1}} G^\alpha = \{\Phi_{z^{-1}} f(t) : f \in G^\alpha\}$. ∎

In the last section of this paper, we prove that the space G^α is always different from the space $\Phi_{z^{-1}} G^\alpha$.

As we said in the Introduction, Question 1 was solved in [D3] for $\alpha = 1$. Indeed, the space G_1^1 is the same as the space S_1^{+0} which was thoroughly studied in [D3] (see the last section for the definition of S_1^{+0}). The dual of the space S_1^{+0} is especially important in relation to the Fourier–Laplace operator (1.1). Indeed, $(S_1^{+0})'$ is the largest space where this operator can be defined so that its range consists of analytic functions on the unit disc. In [D3], we proved that for every analytic function F in D, there exists a unique functional u in $(S_1^{+0})'$ such that $F(w) = \mathscr{F}_D(u)(w) = \langle u(t), e^{-(1/2)((1+w)/(1-w))t} \rangle$.

As the fractional power of the Hankel–Clifford transform $\mathscr{I}_{z,\gamma}$ is an isomorphism from the space $S_1^{+0} = G_1^1$ onto itself, by dualizing, we can extend this transform to the space $(S_1^{+0})'$. We show that there is a close relation between the transform $\mathscr{I}_{z,0}$ and $\mathscr{F}_D$: Indeed, for a functional $u \in (S_1^{+0})'$, the function which we obtain from the analytic function $\mathscr{F}_D(u)(w)$ by doing a rotation of angle ρ in the variable w is almost the same as the function which we obtain by applying the operator $\mathscr{F}_D$ to the functional $\mathcal{I}_{\mathscr{C}^{p},0}(u)$ (see Lemma 3.4 of this paper for a more precise formulation of this relation).

Let us consider the generalized Laguerre orthonormal system in $L^2((0,+\infty), t^\gamma\,dt)$ $(\gamma > -1)$, that is,

$$\mathscr{L}_n^\gamma(t) = \left(\frac{\Gamma(n+\gamma+1)}{n!}\right)^{-1/2} L_n^\gamma(t)\, e^{-t/2},$$

where $L_n^\gamma(t)$ are the Laguerre polynomials.

It is well known that the functions in this orthogonal system are eigenfunctions for the Hankel–Clifford transform, more precisely, $\mathscr{H}_\gamma(\mathscr{L}_n^\gamma(t)) = (-1)^n\,(\Gamma(n+\gamma+1)/n!)^{-1/2}\,\mathscr{L}_n^\gamma(t)$.

Next, we give an estimate of the Laguerre polynomials which we will need to establish a sufficient condition for a sequence to be the sequence of Fourier–Laguerre coefficients of a function in the space G_α^α.

In [D4], we proved the following bounds on the Laguerre polynomials and their derivatives:

If k, p, $n \in \mathbb{N}$, $t \geqslant 0$, and $\gamma \geqslant 0$ then

$$|t^k(L_n^\gamma(t)\, e^{-t/2})^{(p)}| \leqslant 2^{-\min(\gamma,k)} 4^k (n+1)\cdots(n+k) \binom{n+p+\max(\gamma-k,0)}{n} \tag{2.3}$$

and we stated that these bounds are sharp.

However, using the fact that the functions in the Laguerre orthonormal system are eigenfunctions for the Hankel–Clifford transform and the formula (2.2), these bounds can be improved for certain values of k and p.

Indeed, taking in these bounds $k = p/2$, and for $p \geqslant 2\gamma$, we get

$$|t^{p/2}(L_n^\gamma(t)\, e^{-t/2})^{(p)}| \leqslant 2^p (n+1)\cdots\left(n+\left[\frac{p}{2}\right]+1\right)\binom{n+p}{n}.$$

(Here, $[x]$ denotes the integral part of the real number x.) Note that the order in n in these bounds is $n^{3p/2+1}$. We prove that actually, the order in n can be taken to be $n^{p/2+4+\gamma}$.

LEMMA 2.1. *If p, $n \in \mathbb{N}$, $t \geqslant 0$, and $\gamma \geqslant 0$, then*

$$|t^{p/2}(L_n^\gamma(t)\, e^{-t/2})^{(p)}| \leqslant 32(p+4)(n+1)\cdots\left(n+\left[\frac{p}{2}\right]+3\right)(n+1)\binom{n+\gamma}{n}. \tag{2.4}$$

Proof. We first prove the lemma for $\gamma=0$. Indeed, from (2.3), we get

$$\|t^{(p+k)/2}(L_n(t)\,e^{-t/2})^{(k)}\|_2 \leqslant 2^{p+k+5}(n+1)\cdots\left(n+\left[\frac{p+k}{2}\right]+2\right)\binom{n+k}{n}. \tag{2.5}$$

Since $\mathscr{H}_0(L_n(t)\,e^{-t/2})=(-1)^n\,L_n(t)\,e^{-t/2}$, from (2.2) and (2.5), we get

$$\begin{aligned}\|t^{(p+k)/2}(L_n(t)\,e^{-t/2})^{(p)}\|_2 &= \|t^{(p+k)/2}(\mathscr{H}_0(L_n(t)\,e^{-t/2}))^{(p)}\|_2\\ &= 2^{-p+k}\,\|t^{(p+k)/2}(L_n(t)\,e^{-t/2})^{(k)}\|_2\\ &\leqslant 2\cdot 4^{k+2}(n+1)\cdots\left(n+\left[\frac{p+k}{2}\right]+2\right)\binom{n+k}{n}.\end{aligned} \tag{2.6}$$

Hence, using the Hölder inequality, we get

$$|t^{p/2}(L_n(t)\,e^{-t/2})^{(p)}|\leqslant 32(p+4)(n+1)\cdots\left(n+\left[\frac{p}{2}\right]+3\right)(n+1).$$

Now, using

$$L_n^{\gamma}(t)=\sum_{l=0}^{n}\binom{l+\gamma-1}{l}L_{n-l}(t)$$

we have

$$|t^{p/2}(L_n^{\gamma}(t)\,e^{-t/2})^{(p)}| \leqslant 32(p+4)(n+1)\cdots\left(n+\left[\frac{p}{2}\right]+3\right)(n+1)\binom{n+\gamma}{n}$$

and the lemma is proved. ∎

3. Fourier–Laguerre Coefficients in the Space G_α^α

In this section, we characterize the space G_α^α $(\alpha\geqslant 1)$ in terms of the Fourier–Laguerre coefficients. To begin with, we give a sufficient condition on the Fourier–Laguerre coefficients of a function f so that this function is in the space G_α^α.

Lemma 3.1. *Let $f\in L^2(0,+\infty)$ and $\alpha\geqslant 1$; we put $a_n=\int_0^\infty f(t)\,L_n(t)\,e^{-t/2}\,dt$ for its Fourier–Laguerre coefficients. If there exist constants $c>0$ and $a>1$ such that*

$$|a_n|\leqslant c a^{-n^{1/\alpha}} \quad \textit{for} \quad n\geqslant 0 \tag{3.1}$$

then $f\in G_\alpha^\alpha$

Proof. From (2.1), it is enough to prove that the function f belongs to the space S^+ and satisfies

$$\|t^{k/2}f(t)\|_2 \leqslant CA^k k^{\alpha(k/2)} \qquad \text{for} \quad k \geqslant 0$$

and

$$\|t^{p/2}f^{(p)}(t)\|_2 \leqslant CB^p p^{\alpha(p/2)} \qquad \text{for} \quad p \geqslant 0 \tag{3.2}$$

for certain constants A, B, $C > 0$.

As $(a_n)_n$ is a rapidly decreasing sequence, it follows that $f \in S^+$.

Now, taking into account that $\alpha \geqslant 1$, it follows that $a^{-n^{1/\alpha}} \leqslant M^x a^{-(n+x)^{1/\alpha}}$ for $x \geqslant 0$ and for a certain constant $M > 0$ which only depends on a, hence, from (2.5) and (3.1), we get

$$\begin{aligned}
\|t^{k/2}f(t)\|_2 &= \left\| \sum_n a_n t^{k/2} L_n(t)\, e^{-t/2} \right\|_2 \\
&\leqslant \sum_n |a_n| \, \|t^{k/2} L_n(t)\, e^{-t/2}\|_2 \\
&\leqslant c \sum_n a^{-n^{1/\alpha}} 2^k (n+1) \cdots \left(n + \left[\frac{k}{2}\right] + 2\right) \\
&\leqslant c 2^k M^{([k/2]+2)} \sum_n a^{-(n+[k/2]+2)^{1/\alpha}} \left(n + \left[\frac{k}{2}\right] + 2\right)^{([k/2]+2)}.
\end{aligned} \tag{3.3}$$

But the function $\rho_{u,v}(x) = v^{-(x+u)^{1/\alpha}}(x+u)^u$ $(u \geqslant 0, \quad v > 1)$ attains its maximum value on the interval $(-u, +\infty)$ at the point $x = (\alpha u/\log v)^\alpha - u$. Hence, from (3.3) we get

$$\|t^{k/2}f(t)\|_2 \leqslant CA^k k^{(\alpha/2)k}$$

for certain constants C, $A > 0$.

From (2.6), the inequality (3.2) can be proved in the same way. ∎

The rest of this section is devoted to proving that actually, the sufficient condition stated in Lemma 3.1 is also necessary.

To prove this result, we study the behaviour of the Fourier–Laplace operator defined by

$$\mathscr{F}_D(f)(w) = \int_0^\infty f(t)\, e^{-(1/2)((1+w)/(1-w))t}\, dt \qquad \text{for} \quad w \in D \tag{3.4}$$

in the space G_α^α.

Why is it interesting to study this operator? Given a function $f \in L^2(0, +\infty)$, there exists a closed relation between the function $\mathscr{F}_D(f)$

and the Fourier–Laguerre coefficients of f. Indeed, the Fourier–Laguerre coefficients of f are almost the Taylor coefficients of the analytic function $\mathscr{F}_D(f)$. This result is also true for the space $(S_1^{+0})'$ (see Proposition 3.2 of [D3]):

THEOREM C. *Let* $u \in (S_1^{+0})'$ *and* $a_n = \langle u, L_n(t) e^{-t/2} \rangle$. *Then*

$$\mathscr{F}_D(u)(w) = (1-w) \sum_n a_n w^n.$$

In view of this theorem, it will be interesting to have a characterization of the analytic functions on the unit disc whose sequence of Taylor coefficients $(a_n)_n$ satisfies (3.1) for certain constants $c > 0$ and $a > 1$:

LEMMA 3.2. *Let* $F \in H(D)$ *and* $\alpha \geqslant 0$. *If we put* $F(w) = (1-w) \sum_n a_n w^n$, *then the following conditions are equivalent*:

(i) *There exist constants* C, $A > 0$ *such that*

$$|F^{(p)}(w)| \leqslant C A^p p^{\alpha p} \qquad \text{for} \quad p \geqslant 0 \tag{3.5}$$

and for $w \in D$.

(ii) *There exist constants* $c > 0$, $a > 1$ *such that*

$$|a_n| \leqslant c a^{-n^{1/\alpha}} \qquad \textit{for} \quad n \geqslant 0.$$

Proof. We write $F(w) = \sum_n b_n w^n$, where $b_n = a_n - a_{n-1}$ $(a_{-1} = 0)$.

(i) $\to$ (ii). Since $F^{(p)}(w) = \sum_{n \geqslant p} n(n-1) \cdots (n-p+1) b_n w^{n-p}$, from the hypothesis, we get

$$|n(n-1) \cdots (n-p+1) b_n| \leqslant C A^p p^{\alpha p} \qquad \text{for} \quad n \geqslant p.$$

Since $n(n-1) \cdots (n-p+1) \geqslant e^{-p} n^p$ for $n \geqslant p$, we have

$$|b_n| \leqslant C \inf_{p,\, p \leqslant n} \left\{ \frac{(eA)^p\, p^{\alpha p}}{n^p} \right\}. \tag{3.6}$$

We can assume that the constant A is greater than 1. Then, if $p \geqslant n$ the number $(eA)^p\, p^{\alpha p}/n^p$ is greater than $(eA)^n\, n^{\alpha n}/n^n$, and so the infimum in (3.6) can be taken varying on $p \geqslant 0$. Thus, from (2) and (3) in [GS1, pp. 169–170], we have

$$|b_n| \leqslant C \inf_{p \geqslant 0} \left\{ \frac{(eA)^p\, p^{\alpha p}}{n^p} \right\} \leqslant c a^{-n^{1/\alpha}}$$

for certain constants $c > 0$ and $a > 1$. Now, (i) → (ii) follows because

$$a_n = \sum_{k=0}^{n} b_k.$$

(ii) → (i). Indeed, since $F^{(p)}(w) = \sum_{n \geq p} n(n-1) \cdots (n-p+1)(a_n - a_{n-1}) w^{n-p}$, we get

$$|F^{(p)}(w)| \leqslant 2C \sum_n n^p a^{-n^{1/\alpha}}.$$

But the function $\rho(x) = x^p u^{-x^{1/\alpha}}$ $(u > 1)$ attains its maximum value on the interval $(0, +\infty)$ at the point $x = (\alpha p/\log u)^\alpha$. Now, it is easy to finish the proof. ∎

Hence, we must prove that if a function f belongs to the space G_α^α, then the analytic function $\mathscr{F}_D(f)$ satisfies part(i) of the previous lemma.

The following theorem is a first step in this proof.

THEOREM 3.3. *Let $f \in G_\alpha$ and $\alpha \geqslant 0$. Then there exist constants C, $A > 0$ such that if $w \in D$ and $\Re w \leqslant 0$, then*

$$|(\mathscr{F}_D(f))^{(p)}(w)| \leqslant C A^p p^{\alpha p}$$

for all $p \geqslant 0$.

Proof. From (3.4) and using the chain rule for higher derivatives (see [Sc, p. 12]), we get

$$\begin{aligned}(\mathscr{F}_D)^{(p)}(f)(w) = &\sum_{k=1}^{p} \frac{(-1)^k}{k!} \sum_{m=1}^{k} (-1)^m \binom{k}{m} \\ &\times \left(\frac{1}{2}\frac{1+w}{1-w}\right)^{k-m} \left(\left(\frac{1}{2}\frac{1+w}{1-w}\right)^m\right)^{(p)} \\ &\times \int_0^\infty (-t)^k f(t)\, e^{-(1/2)((1+w)/(1-w))t}\, dt. \end{aligned} \tag{3.7}$$

For $|w| \leqslant 1$ and $\Re w \leqslant 0$, we have

$$\left|\frac{1+w}{1-w}\right| \leqslant 1.$$

By induction on m, it can be proved that

$$\left|\left(\left(\frac{1+w}{1-w}\right)^m\right)^{(p)}\right| \leqslant \frac{2^m m \cdots (m+p-1)}{|1-w|^{m+p}} \qquad \text{for} \quad w \in D. \tag{3.8}$$

Hence, for $|w| \leqslant 1$ and $\Re w \leqslant 0$, we get

$$\left|\left(\left(\frac{1}{2}\,\frac{1+w}{1-w}\right)^m\right)^{(p)}\right| \leqslant \frac{m\cdots(m+p-1)}{|1-w|^{m+p}} \leqslant m\cdots(m+p-1).$$

Also, since $f \in G_\alpha$, we get $|t^k f(t)| \leqslant MB^k k^{\alpha k}$ for certain constants M, $B > 0$. Hence, we have

$$\left|\int_0^\infty (-t)^k\, f(t)\, e^{(1/2)((1+w)/(1-w))t}\, dt\right| \leqslant M'(B')^k\, k^{\alpha k} \qquad \text{for} \quad w \in D$$

and certain constants M', $B' > 0$.

From (3.7) and (3.8), we obtain

$$\begin{aligned} |(\mathscr{F}_D)^{(p)}\,(f)(w)| &\leqslant \sum_{k=1}^{p} \frac{1}{k!} \sum_{m=1}^{k} \binom{k}{m} m\cdots(m+p-1)\, M'(B')^k\, k^{\alpha k} \\ &\leqslant M'\,\frac{(2p)!}{p!} \sum_{k=1}^{p} \frac{1}{k!}\,(2B')^k\, k^{\alpha k} \\ &\leqslant CA^p p^{\alpha p} \end{aligned}$$

for certain constants C, $A > 0$. ∎

Let us consider a function f in the space $G_\alpha^\alpha = G_\alpha \cap G^\alpha$. Using the fact that the function belongs to the space G_α, from Theorem 3.3, we get that on the left side of the unit disc, the function $\mathscr{F}_D(f)$ satisfies

$$|(\mathscr{F}_D(f))^{(p)}\,(w)| \leqslant CA^p p^{\alpha p} \qquad \text{for} \quad p \geqslant 0. \tag{3.9}$$

But, what happens on the right side of the unit disc?. In other words, what is the analytic function $\mathscr{F}_D(f)(-w)$ on the left side of the unit disc like?

Let us consider a related question which will solve the previous one: Given a functional $u \in (S_1^{+0})'$, and its associated analytic function $\mathscr{F}_D(u)$, consider the analytic function g which we obtain from $\mathscr{F}_D(u)$ by doing a rotation in the variable, that is, $g(w) = \mathscr{F}_D(u)(z_0 w)$, when z_0 is a fixed complex number of modulus equal to 1. As we said above, there exists another functional $v \in (S_1^{+0})'$ such that $g(w) = \mathscr{F}_D(v)(w)$. The question is: What is the relation between u and v like? As we said in Section 2, the fractional power of the Hankel–Clifford transform $\mathscr{I}_{z_0,0}$ plays an important role in answering this question:

Lemma 3.4. *Let $u \in (S_1^{+0})'$. Let us consider the analytic function $g(w) = \mathscr{F}_D(u)(z_0 w)$ for z_0 satisfying $|z_0| = 1$ and $z_0 \neq 1$. Then if we put*

$$v = \left(\left(\frac{1+z_0}{2}\right)\delta + (1-z_0)\,\delta'\right) * (\mathscr{I}_{z_0,0}(u))$$

we get $g(w) = \mathscr{F}_D(v)(w)$ for $w \in D$.

Proof. From Theorem C, we get for the function g the expression

$$g(w) = (1 - z_0 w) \sum_n a_n z_0^n w^n,$$

where $a_n = \langle u, L_n(t)\, e^{-t/2} \rangle$.

From Corollary 4.7 in [D3], we get

$$\mathscr{F}_D(v) = \mathscr{F}_D\left(\left(\frac{1+z_0}{2}\right)\delta + (1-z_0)\,\delta'\right)\mathscr{F}_D(\mathscr{I}_{z_0,0}(u)).$$

It is straightforward that

$$\mathscr{F}_D\left(\left(\frac{1+z_0}{2}\right)\delta + (1-z_0)\,\delta'\right) = \frac{1-z_0 w}{1-w}.$$

Since $\mathscr{I}_{z_0,0}(L_n(t)\, e^{-t/2}) = z_0^n L_n(t)\, e^{-t/2}$, we get $\langle \mathscr{I}_{z_0,0}(u), L_n(t)\, e^{-t/2} \rangle = z_0^n a_n$, and again from Theorem C, it follows that

$$\mathscr{F}_D(\mathscr{I}_{z_0,0}(u)) = (1-w) \sum_n a_n z_0^n w^n.$$

Hence, the lemma is proved. ∎

Now, we return to our question: What is the analytic function $\mathscr{F}_D(f)(-w)$ on the left side of the unit disc like ? Note that from the previous lemma, we have that the analytic function $\mathscr{F}_D(f)(-w)$ is very related to the analytic function $\mathscr{F}_D(\mathscr{H}_0(f))(w)$ (because of the equality $\mathscr{I}_{-1,0} = \mathscr{H}_0$). This relation is the key to stating the next corollary, where we prove that also on the right side of the unit disc, the analytic function $\mathscr{F}_D(f)(w)$ (for $f \in G_\alpha^\alpha$) satisfies (3.9):

COROLLARY 3.5. *Let $f \in G_\alpha^\alpha$ and $\alpha \geqslant 1$. Then there exist constants C, $A > 0$ such that*

$$|(\mathscr{F}_D(f))^{(p)}(w)| \leqslant C A^p p^{\alpha p} \qquad \text{for} \quad p \geqslant 0 \text{ and } w \in D$$

and $\lim_{w \to 1} \mathscr{F}_D(f)(w) = 0$.

Proof. First of all, as $f \in S^+$, from Theorem C, it follows that $\lim_{w \to 1} \mathscr{F}_D(f)(w) = 0$.

Since $G_\alpha^\alpha = G_\alpha \cap G^\alpha$, the hypothesis is equivalent to $f \in G_\alpha$ and $f \in G^\alpha$. Hence, by Theorem 3.3, we have proved the thesis when $w \in D$ and $\Re w \leqslant 0$.

Let us consider the function $h(t) = \mathscr{H}'_0(f)(t)$ and the functional $g = 2\delta' * \mathscr{H}_0(f) = 2\mathscr{H}_0(f)(0) \cdot \delta + \mathscr{H}'_0(f)(t)$.

From Lemma 3.4, we get

$$\mathscr{F}_D(f)(-w) = \mathscr{F}_D(g)(w) = 2\mathscr{H}_0(f)(0) + \mathscr{F}_D(h)(w).$$

As $f \in G^\alpha$, from Theorem A, it follows that $\mathscr{H}_0 f \in G_\alpha$ and since G_α is closed under differentiation, we get $h \in G_\alpha$. Again, by applying Theorem 3.3, the proof is finished. ∎

Now, Lemma 3.1, Corollary 3.5, Theorem C, and Lemma 3.2 give the main result in this paper:

THEOREM 3.6. *Let $f \in L^2(0, +\infty)$, $\alpha \geq 1$, $a_n = \int_0^\infty f(t)\, L_n(t)\, e^{-t/2}\, dt$, and $\mathscr{F}_D(f)(w)$ be as in (3.4). The following conditions are equivalent:*

(i) *There exist two constants $c > 0$ and $a > 1$ such that*

$$|a_n| \leqslant c a^{-n^{1/\alpha}} \qquad \text{for} \quad n \geqslant 0.$$

(ii) *The function f belongs to the space G_α^α.*

(iii) *There exist constants C, $A > 0$ such that*

$$|\mathscr{F}_D^{(p)}(f)(w)| \leqslant C A^p p^{\alpha p} \qquad \text{for} \quad p \geqslant 0 \text{ and } w \in D$$

and $\lim_{w \to 1} \mathscr{F}_D(f)(w) = 0$.

Conversely, given a sequence $(a_n)_n$ satisfying condition (i) or given an analytic function F on the unit disc satisfying the bounds (3.5) and $\lim_{w \to 1} F(w) = 0$, there exists $f \in G_\alpha^\alpha$ such that $a_n = \int_0^\infty f(t)\, L_n(t)\, e^{-t/2}\, dt$ for $n \in \mathbb{N}$ and $\mathscr{F}_D(f)(w) = F(w)$ for $w \in D$.

Now, we extend condition (i) of the previous theorem for the generalized Laguerre polynomials.

Indeed, for $\gamma > -1$, the sequence

$$a_n^{(\gamma)} = \left(\frac{\Gamma(n+\gamma+1)}{n!} \right)^{-1/2} \int_0^\infty f(t)\, t^\gamma L_n^\gamma(t)\, e^{-t/2}\, dt \tag{3.10}$$

is called the γ Fourier–Laguerre coefficients of f.

COROLLARY 3.7. *Let $f \in L^2(0, +\infty)$, $\gamma \geqslant 0$, and $(a_n^{(\gamma)})_n$ be as in (3.10). Then $f \in G_\alpha^\alpha$ ($\alpha \geqslant 1$) if and only if there exist two constants $c > 0$, $a > 1$ (which depend on γ) such that $|a_n^{(\gamma)}| \leqslant c a^{-n^{1/\alpha}}$ for all $n \geqslant 0$.*

Proof. Because $\sum_k |\binom{\gamma}{k}| < \infty$ for any $\gamma \geqslant 0$, the proof is the same as that of Theorem 2.4 in [D3]. ∎

Let us consider the sequence space

$$\{(a_n)_n : \exists a > 1,\ \|(a_n)_n\|_a = \sup_n \{|a_n a^{n^{1/\alpha}}|\} < \infty\}.$$

We can endow this space with an inductive limit topology. Hence, since the canonical sequence $(e_{m,n})_n$ defined by

$$e_{m,n} = \delta_{n,m}$$

is a basis of that sequence space, using the Closed Graph Theorem for LF-spaces, we get that the expansion for every function f in the space G_α^α converges for the topology of G_α^α.

To finish this section, we generalize Theorem 3.6 for the dual spaces $(G_\alpha^\alpha)'$ $(\alpha \geq 1)$.

Since the functions $\mathscr{L}_n^\gamma(t)$, $e^{-(1/2)((1+w)/(1-w))t} \in G_\alpha^\alpha$ for $\alpha \geq 1$, $\gamma \geq 0$, and $w \in D$, we can define the operators $\mathscr{L}_\gamma: (G_\alpha^\alpha)' \to \mathbb{C}^\mathbb{N}$ and $\mathscr{F}_D: (G_\alpha^\alpha)' \to H(D)$ by

$$\mathscr{L}_\gamma(u) = (\langle u, \mathscr{L}_n^\gamma(t) \rangle)_n$$

$$\mathscr{F}_D(u)(w) = \langle u(t), e^{-(1/2)((1+w)/(1-w))t} \rangle \qquad \text{for} \quad w \in D.$$

Hence, dualizing parts (i) and (ii) in Theorem 3.6, we get

COROLLARY 3.8. *The mapping*

$$\mathscr{L}_\gamma: (G_\alpha^\alpha)' \to \{(a_n)_n : \forall a > 1,\ \|(a_n)_n\|_a = \sup_n \{|a_n a^{-n^{1/\alpha}}|\} < \infty\}$$

is an isomorphism between these spaces. (In the space $(G_\alpha^\alpha)'$ we consider the strong topology and in the sequence space, the Fréchet topology generated by the seminorms $\|\cdot\|_a$ for $a > 1$.)

Finally, we characterize the range of the operator $\mathscr{F}_D$ acting on $(G_\alpha^\alpha)'$.

COROLLARY 3.9. *Let $H(\alpha, D)$ $(\alpha \geq 1)$ be the following subspace of analytic functions on the unit disc,*

$$H(\alpha, D) = \{F \in H(D) : \forall \varepsilon > 0,$$
$$\|F\|_\varepsilon = \sup\{|F(w)|\, e^{-\varepsilon(1/(1-|w|))^{1/(\alpha-1)}} : w \in D\} < \infty\},$$

endowed with the Fréchet topology generated by the seminorms $\|\cdot\|_\varepsilon$, $\varepsilon > 0$. Then, the mapping

$$\mathscr{F}_D: (G_\alpha^\alpha)' \to H(\alpha, D)$$

is an isomorphism between these spaces (as in the previous corollary, in the space $(G_\alpha^\alpha)'$ we consider the strong topology).

Proof. By using Corollary 3.8 and Theorem C, it will be enough to prove that an analytic function F belongs to the space $H(\alpha, D)$ if and only if its Taylor sequence $(a_n)_n$ belongs to the sequence space

$$\{(a_n)_n : \forall a > 1,\ \|(a_n)_n\|_a = \sup_n \{|a_n a^{-n^{1/\alpha}}|\} < \infty\}.$$

($\Rightarrow$) Let us write $F(w) = \sum_n a_n w^n$. Since $F \in H(\alpha, D)$ and by using the Cauchy formula, we get for every $\varepsilon > 0$ a constant $c_\varepsilon > 0$ such that

$$|a_n| \leqslant c_\varepsilon \left(\frac{1}{r}\right)^n e^{\varepsilon(1/(1-r))^{1/(\alpha-1)}} \qquad \text{for} \quad 0 < r < 1 \text{ and } n \in \mathbb{N}. \tag{3.11}$$

From (3) of [GS1, p. 170], it follows that

$$e^{\varepsilon(1/(1-r))^{1/(\alpha-1)}} \leqslant C \inf_k{}^{-1} \left\{ \frac{k^{(\alpha-1)k}(1-r)^k}{d_\varepsilon^k} \right\} \tag{3.12}$$

for certain constant $C > 0$, which does not depend on ε, r, and $d_\varepsilon > 0$, which does not depend on r.

Taking $r = n/(n+k)$, we get $r^n(1-r)^k \geqslant e^{-k}(k^k/n^k)$. Hence, from (3.11) and (3.12), we have

$$|a_n| \leqslant c_\varepsilon \inf_k{}^{-1} \{k^{\alpha k}/(ed_\varepsilon n)^k\}.$$

Again, applying (3) of [GS1, p. 170], we conclude that

$$|a_n| \leqslant C_\varepsilon e^{\varepsilon(en)^{1/\alpha}}.$$

($\Leftarrow$) From the hypothesis, for all $a > 1$ we get

$$|F(w)| \leqslant c_a \sum_n a^{n^{1/a}} |w|^n \leqslant C_a \sup_n \{a^{2n^{1/\alpha}} |w|^n\} \qquad \text{for} \quad w \in D. \tag{3.13}$$

From (3) of [GS1, p. 170], it follows that

$$a^{2n^{1/\alpha}} |w|^n = e^{(2 \log a) n^{1/\alpha}} |w|^n \leqslant C\, |w|^n \inf_k{}^{-1} \left\{ \frac{k^{\alpha k} n^k}{d_a^k} \right\} \tag{3.14}$$

for certain constants $C > 0$, which does not depend on a, n, and $d_a > 0$, which does not depend on n.

Since

$$\begin{aligned} |w|^n n^{-k} &\leqslant \left(\frac{1}{k} \log\left(\frac{1}{|w|}\right)\right)^{-k} \\ &\leqslant C \left(\frac{1}{k(1-|w|)}\right)^{-k} \qquad \text{for all} \quad n \geqslant 0, \end{aligned}$$

from (3.13) and (3.14), we get

$$|F(w)| \leqslant C_a \inf_k{}^{-1} \left\{ \frac{k^{(\alpha-1)k}(1-|w|)^{-k}}{d_a^k} \right\}.$$

Again applying (3) of GS1, p. 170], we conclude that

$$|F(w)| \leqslant C_a e^{(2 \log a)(1/(1-|w|))^{1/(\alpha-1)}}. \qquad \blacksquare$$

4. Appendix

In Theorem 2.6 of [D2], we stated the relation between the spaces G^{β} and $S^{+\beta}$, that is, for $\beta \geqslant 1$, the space G^{β} is contained in the space $S^{+\beta-1}$. Here, we show that for every $\beta \geqslant 1$, the result $G^{\beta} \subset S^{+\beta-1}$ is the best possible, that is, G^{β} is always different from the space $S^{+\beta-1}$, and $S^{+\beta-1}$ is the smallest space of type $S^{+\gamma}$, which contains the space G^{β}. To prove this result, we give an estimate for the analytic function $\mathscr{F}_D(f)$, when $f \in S^{+\beta}$ or when $f \in G^{\beta}$. These estimates are similar to those given in Theorem 3.3 for the functions in the space G_{α}.

To begin with, we recall the definition of the Gel'fand–Shilov spaces S_{α}^{+}, $S^{+\beta}$, and $S_{\alpha}^{+\beta}$, defined on $(0, +\infty)$.

$$S_{\alpha}^{+} = \{ f \in \mathscr{C}^{\infty}((0, +\infty)) : \exists A,\ C_p > 0,\ \forall k,\ p \geqslant 0,\ \sup_{t > 0} |t^k f^{(p)}(t)| \leqslant C_p A^k k^{\alpha k} \}$$

$$S^{+\beta} = \{ f \in \mathscr{C}^{\infty}((0, +\infty)) : \exists B,\ C_k > 0,\ \forall k,\ p \geqslant 0,\ \sup_{t > 0} |t^k f^{(p)}(t)| \leqslant C_k B^p p^{\alpha p} \}$$

$$S_{\alpha}^{+\beta} = \{ f \in \mathscr{C}^{\infty}((0, +\infty)) : \exists A,\ B,\ C > 0,\ \forall k,\ p \geqslant 0,\ \sup_{t > 0} |t^k f^{(p)}(t)| \leqslant C A^k B^p k^{\alpha k} p^{\beta p} \}.$$

A careful study of the proof of the Theorem 3.3 shows that the condition $\Re w \leqslant 0$ can be slightly improved:

Corollary 4.1. *Let $f \in G_{\alpha}$ and $\alpha \geqslant 0$. Then for every θ, $-1 < \theta < 1$, there exist constants C, $A > 0$, which depend on θ such that if $w \in D$ and $\Re w \leqslant \theta$, then*

$$|(\mathscr{F}_D(f))^{(p)}(w)| \leqslant C A^p p^{\alpha p}$$

for all $p \geqslant 0$.

Using Lemma 3.4, we can give an analogous estimate for the functions in the space G^{β}.

Corollary 4.2. *Let $f \in G^{\beta}$ and $\beta \geqslant 0$. Then for every θ, $-1 < \theta < 1$, there exist constants C, $A > 0$, which depend on θ such that if $w \in D$ and $\Re w \geqslant \theta$, then*

$$|(\mathscr{F}_D(f))^{(p)}(w)| \leqslant C A^p p^{\beta p}$$

for all $p \geqslant 0$.

Proof. Given a function $f \in G^{\beta}$, consider the functions $g(t) = ((2\delta') * (\mathscr{H}_0(f)))(t)$, and $h(t) = \mathscr{H}'_0(f)(t)$, and proceed as in the proof of Corollary 3.5. ∎

Now, we extend the estimate in the previous corollaries for the functions in the space $S^{+\beta}$. To prove it, we use the Cauchy formula. As a consequence, this estimate is stated for the complex numbers in a certain Stolz angle contained inside the unit disc. It should be noted that the estimates for the functions in the space G^{β} were proved for the complex numbers in a half-disc. This difference is the key to proving that the result $G^{\beta} \subset S^{+\beta-1}$ is the best possible.

THEOREM 4.3. *Let $f \in S^{+\beta}$, $\beta \geq 0$, and $\Delta_{\theta,\eta}$ $(-1 < \theta < 1,\ 1 < \eta)$ be the portion of the Stolz angle contained in the half-plane $\Re w \geq \theta$, that is,*

$$\Delta_{\theta,\eta} = \left\{ w \in D : \Re w \geq \theta,\ \frac{|1-w|}{1-|w|} < \eta \right\}.$$

Then there exist constants C, $A > 0$ (which depend on θ, η) such that

$$|(\mathscr{F}_D(f))^{(p)}(w)| \leq C A^p p^{(\beta+1)p} \qquad \text{for} \quad p \geq 0 \text{ and } w \in \Delta_{\theta,\eta}.$$

(By continuity, these bounds remain true for $w = 1$.)

Proof. As $f \in S^{+\beta}$, we get

$$|f^{(p)}(t)| \leq M B^p p^{\beta p} \tag{4.1}$$

for certain constants M, $B > 0$. Integrating by parts in the expression of the function $\mathscr{F}_D(f)$, we have

$$\mathscr{F}_D(f)(w) = \sum_{m=0}^{p-1} f^{(m)}(0) \left(2\,\frac{1-w}{1+w}\right)^{m+1} + \left(2\,\frac{1-w}{1+w}\right)^{p} \int_0^{\infty} f^{(p)}(t)\, e^{-(1/2)((1+w)/(1-w))t}\, dt.$$

We put

$$h(w) = \sum_{m=0}^{p-1} f^{(m)}(0) \left(2\,\frac{1-w}{1+w}\right)^{m+1}$$

and

$$g(w) = \left(2\,\frac{1-w}{1+w}\right)^{p} \int_0^{\infty} f^{(p)}(t)\, e^{-(1/2)((1+w)/(1-w))t}\, dt. \tag{4.2}$$

If $\Re w \geqslant \theta$, from (4.1) and (3.8), we get

$$
\begin{aligned}
|h^{(p)}(w)| &\leqslant \sum_{m=0}^{p-1} \left| f^{(m)}(0) \left(\left(2\,\frac{1-w}{1+w} \right)^{m+1} \right)^{(p)} \right| \\
&\leqslant \sum_{m=0}^{p-1} \frac{4M(4B)^m\, m^{\beta m}(m+1)\cdots(m+p)}{|1+w|^{m+p+1}} \\
&\leqslant C_\theta \sum_{m=0}^{p-1} (4B)^m\, m^{\beta m}(m+p)^p \\
&\leqslant CA^p p^{(\beta+1)p}
\end{aligned}
$$

for certain constants C, $A > 0$ depending on θ.

From (4.1) and (4.2), it is straightforward that

$$
|g(w)| \leqslant C' \left(\frac{|1-w|}{|1+w|} \right)^p (B')^p\, p^{\beta p} \tag{4.3}
$$

for certain constants C', $B' > 0$.

To estimate the derivatives of the function g, we use the Cauchy formula.

For every $w \in \Delta_{\theta,\eta}$, we put Γ_w for the circle with center at w and radius $r = |1-w|/\eta\rho_\theta$, where $\rho_\theta = \max\{1, 2((1-\theta)/(1+\theta))\}$. From the definition of the Stolz angle, it follows that Γ_w is contained into D. For $\zeta \in \Gamma_w$, we get

$$
\begin{aligned}
&\frac{|1-\zeta|}{|w-\zeta|} \leqslant \frac{r + |1-w|}{r} \leqslant 1 + \eta\rho_\theta \\
&\Re\zeta \geqslant \Re w - \frac{|1-w|}{\eta\rho_\theta} \geqslant \Re w - \frac{1+\theta}{2(1-\theta)}\,(1-|w|) \geqslant \frac{\theta-1}{2}.
\end{aligned} \tag{4.4}
$$

Hence, from the Cauchy formula, (4.3), and (4.4), we get

$$
\begin{aligned}
|g^{(p)}(w)| &= \left| \frac{p!}{2\pi i} \int_{\Gamma_w} \frac{g(\zeta)}{(w-\zeta)^{p+1}}\, d\zeta \right| \\
&\leqslant \frac{p!}{2\pi} \int_{\Gamma_w} \left| \frac{g(\zeta)}{(w-\zeta)^{p+1}} \right| |d\zeta| \\
&\leqslant C'(B')^p\, p^{\beta p}\, \frac{p!}{2\pi} \int_{\Gamma_w} \left(\frac{1}{|1+\zeta|} \right)^p \left(\left| \frac{1-\zeta}{w-\zeta} \right| \right)^p \frac{1}{|w-\zeta|}\, |d\zeta| \\
&\leqslant C'(C_\theta(1+\rho_\theta\eta)\, B')^p\, p^{(\beta+1)p}
\end{aligned}
$$

and the theorem is proved. ∎

Taking into account that $S_\alpha^{+\alpha-1} \subset S^{+\alpha-1} \cap S_\alpha^\alpha = S^{+\alpha-1} \cap G_\alpha$, we obtain from Corollary 4.1 and Theorem 4.3

COROLLARY 4.4. *Let* $f \in S_\alpha^{+\alpha-1}$, $\alpha \geq 1$, *and* $\mathcal{S}_\theta$ $(0 < \theta < \pi/2)$ *be the portion of the symmetric angle with vertex in* 1 *and amplitude* θ *contained in the unit disc, that is,*

$$\mathcal{S}_\theta = \{w \in D : |\arg(1-w)| \leq \theta\}.$$

Then there exist constants C, $A > 0$ (*which depend on* θ) *such that*

$$|(\mathcal{F}_D(f))^{(p)}(w)| \leq CA^p p^{\alpha p} \qquad \text{for} \quad p \geq 0 \text{ and } w \in \mathcal{S}_\theta.$$

The difference between this corollary and Corollary 3.5 should be noted. Here, we are able to prove the estimate only for the complex numbers in an angle. If this estimate were proved for all complex numbers in the unit disc, we would get the equality between the spaces G_α^α and $S_\alpha^{+\alpha-1}$.

Now, we are ready to state the main results in this section. First, for $\beta \geq 1$, we prove that $G^\beta \neq S^{+\beta-1}$.

LEMMA 4.5. *For every* $\beta \geq 1$, *there exists a function* $f \in S^{+\beta-1}$ *which does not belong to* G_β.

Proof. Indeed, let g be a $\mathcal{C}^\infty$ function with support contained in $[0, 1]$ and satisfying

$$g^{(p)}(\tfrac{1}{2}) = p^{(\beta/2+1)p} \qquad \text{for} \quad p \geq 0 \tag{4.5}$$

and

$$\int_0^1 e^{-(i/2)x} g(x)\, dx = 0. \tag{4.6}$$

Since g has compact support, it follows that $g \in G_\beta$. But from (4.5), we get $g \notin G^\beta$. Hence, $g \in G_\beta$ and $g \notin G_\beta^\beta$. From Corollaries 4.1 and 3.5, there exist two sequences $(w_n)_n$, $(p_n)_n$, such that $(w_n)_n$ is a sequence of complex numbers in the unit disc, with $\lim_{n\to\infty} w_n = 1$, and $(p_n)_n$ is a sequence of nonnegative integers for which

$$|(\mathcal{F}_D(g))^{(p_n)}(w_n)| \geq n n^{p_n} p_n^{\beta p_n}. \tag{4.7}$$

We put

$$f = \left(\left(\frac{1+i}{2}\right)\delta + (1-i)\,\delta'\right) * (\mathcal{I}_{i,0}(g)). \tag{4.8}$$

From (4.6), it follows that $(\mathcal{I}_{i,0}(g))(0) = 0$, and hence, we get

$$f(t) = \left(\frac{1+i}{2}\right)(\mathcal{I}_{i,0}(g))(t) + (1-i)(\mathcal{I}_{i,0}(g))'(t).$$

Since $g \in G_\beta$, from Theorem B, we get that $f \in e^{(i/2)t} G^\beta$. As $G^\beta \subset S^{+\beta-1}$ and $e^{(i/2)t} S^{+\beta-1} \subset S^{+\beta-1}$, we have $f \in S^{+\beta-1}$.

Now, we see that $f \notin G^\beta$.

From Lemma 3.4, we get $\mathscr{F}_D(f)(w) = \mathscr{F}_D(g)(iw)$, hence, (4.7) gives

$$|(\mathscr{F}_D(f))^{(p_n)}(iw_n)| \geqslant n n^{p_n} p_n^{\beta p_n}.$$

So, $\mathscr{F}_D(f)(w)$ cannot satisfy Corollary 4.2 for $\theta = -\frac{1}{2}$. So, the lemma is proved. ∎

It should be noted that in the previous proof the function f in (4.8) can be changed to f_z defined by

$$f_z = \left(\left(\frac{1+z}{2}\right)\delta + (1-z)\,\delta'\right) * (\mathscr{I}_{z,0}(g))$$

for z any complex number in the unit circle different from 1. From Theorem B, the function f_z satisfies $f_z \in e^{(1/2)((1+z)/(1-z))t} G^\beta$, but $f_z \notin G^\beta$. Thus, we have proved that for any $\beta \geqslant 1$ and any complex number z in the unit circle different from 1, the space G^β is not closed under multiplication by $e^{(1/2)((1+z)/(1-z))t}$.

Now, we prove that $S^{+\beta-1}$ is the smallest space of type $S^{+\gamma}$, which contains the space G^β.

LEMMA 4.6. *Let $\beta \geqslant 1$. Then for every $\gamma < \beta$ there exists a function $f \in G^\beta$ which does not belong to $S^{+\gamma-1}$.*

Proof. Let $a > 1$, and $a_0 = -\sum_{n \geqslant 1} a^{-n^{1/\beta}}$. Let us consider the function $F(w) = a_0 + \sum_{n \geqslant 1} a^{-n^{1/\beta}} w^n$. We can write $F(w) = (1-w)\sum_n (\sum_{k=n+1}^{\infty} a^{-n^{1/\beta}}) w^n$. Hence, by Lemma 3.2 and Theorem 3.6, there exists a function $f \in G^\beta_\beta$ such that $F = \mathscr{F}_D(f)$. So, $f \in G^\beta$.

Given $\gamma < \beta$, from Theorem 4.3, to prove that $f \notin S^{+\gamma-1}$, it is enough to prove that there exist constant C, $A > 0$ such that

$$|F^{(p)}(1)| \geqslant C A^p p^{\beta p}.$$

But

$$\begin{aligned} |F^{(p)}(1)| &= \sum_{n \geqslant p} n(n-1)\cdots(n-p+1)\, a^{-n^{1/\beta}} \\ &\geqslant \sum_{n \geqslant p} (n-p)^p\, a^{-n^{1/\beta}} \\ &\geqslant (n-p)^p\, a^{-n^{1/\beta}} \end{aligned}$$

for all n such that $n \geqslant p$. Taking $n = [p^\beta] + p$, the lemma is proved. ∎

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